

## Some classes of Lorentzian $\alpha$ -Sasakian manifolds with respect to quarter-symmetric metric connection

Santu Dey<sup>1</sup>, Buddhadev Pal<sup>2</sup> and Arindam Bhattacharyya<sup>3</sup>

<sup>1,3</sup>Department of Mathematics, Jadavpur University, Kolkata-700032, India

<sup>2</sup>Department of Mathematics, Institute of Sciences, Banaras Hindu University, Varanasi, Uttar Pradesh 221005

E-mail: santu.mathju@gmail.com<sup>1</sup>, pal.buddha@gmail.com<sup>2</sup>, bhattachar1968@yahoo.co.in<sup>3</sup>

### Abstract

The object of the present paper is to study a quarter-symmetric metric connection in a Lorentzian  $\alpha$ -Sasakian manifold. We study some curvature properties of Lorentzian  $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection. We investigate quasi-projectively flat,  $\varphi$ -symmetric,  $\varphi$ -projectively flat Lorentzian  $\alpha$ -Sasakian manifolds with respect to quarter-symmetric metric connection. We also discuss Lorentzian  $\alpha$ -Sasakian manifold admitting quarter-symmetric metric connection satisfying  $\tilde{P}\tilde{S} = 0$ , where  $\tilde{P}$  denote the projective curvature tensor with respect to quarter-symmetric metric connection.

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## 1 Introduction

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten ([4]). Further, Hayden ([6]), introduced the idea of metric connection with torsion on a Riemannian manifold. In ([13]), Yano studied some curvature conditions for semi-symmetric connections in Riemannian manifolds. In 1975, Golab ([5]) defined and studied quarter-symmetric connection in a differentiable manifold.

A linear connection  $\tilde{\nabla}$  on an  $n$ -dimensional Riemannian manifold  $(M^n, g)$  is said to be a quarter-symmetric connection ([5]) if its torsion tensor  $\tilde{T}$  defined by

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \tag{1.1}$$

is of the form

$$\tilde{T}(X, Y) = \eta(Y)\varphi X - \eta(X)\varphi Y, \tag{1.2}$$

where  $\eta$  is 1-form and  $\varphi$  is a tensor field of type  $(1, 1)$ . In addition, if a quarter-symmetric linear connection  $\tilde{\nabla}$  satisfies the condition

$$(\tilde{\nabla}_X g)(Y, Z) = 0 \tag{1.3}$$

for all  $X, Y, Z \in \chi(M)$ , where  $\chi(M)$  is the set of all differentiable vector fields on  $M^n$ , then  $\tilde{\nabla}$  is said to be quarter-symmetric metric connection. In particular, if  $\varphi X = X$  and  $\varphi Y = Y \forall X, Y \in \chi(M)$ ,

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then the quarter-symmetric connection reduces to a semi-symmetric connection ([4]).

In 1980, R. S. Mishra and S. N. Pandey ([7]) studied quarter-symmetric metric connection and in particular, Ricci quarter-symmetric symmetric metric connection on Riemannian, Sasakian and Kaehlerian manifolds. Note that a quarter-symmetric metric connection is a Hayden connection with the torsion tensor of the form (1, 2). Studies of various types of quarter-symmetric metric connection and their properties by various authors in ([10], [11]), ([1]) and ([14]) among others.

The notion of locally symmetry of Riemannian manifolds have been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi ([12]) introduced the notion of locally  $\varphi$ -symmetry on Sasakian manifolds. In the context of contact geometry the notion of  $\varphi$ -symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke ([2]) with several examples.

In 2005, Yildiz and Murathan ([15]) studied Lorentzian  $\alpha$ -Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian  $\alpha$ -Sasakian manifolds are locally isometric with a sphere. In 2012, Yadav and Suthar ([16]) studied Lorentzian  $\alpha$ -Sasakian manifolds. In 2015, Dey and Bhattacharyya [3] studied Lorentzian  $\alpha$ -Sasakian manifolds with respect to quarter-symmetric metric connection.

**Definition 1.1.** A Lorentzian  $\alpha$ -Sasakian manifold  $M^n$  is said to be quasi-projectively flat if

$$g(P(\varphi X, Y)Z, \varphi W) = 0 \quad (1.4)$$

for all vector fields  $X, Y, Z, W$  orthogonal to  $\xi$ .

**Definition 1.2.** A Lorentzian  $\alpha$ -Sasakian manifold  $M^n$  is said to be  $\varphi$ -symmetric if

$$\varphi^2((\nabla_W R)(X, Y)Z) = 0 \quad (1.5)$$

for arbitrary vector fields  $X, Y, Z, W$ .

The Projective curvature tensor is an important tensor from the differential geometric point of view. Let  $M$  be a  $n$ -dimensional Riemannian manifold. If there exists a one-to-one correspondence between each co-ordinate neighbourhood of  $M$  and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then  $M$  is said to be locally projectively flat. For  $n \geq 3$ ,  $M$  is locally projectively flat if and only if the projective curvature tensor vanishes. Here the projective curvature tensor  $P$  is defined by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y] \quad (1.6)$$

for  $X, Y, Z \in \chi(M)$ , where  $S$  is the Ricci tensor of the manifold. In fact  $M$  is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

**Definition 1.3.** A Lorentzian  $\alpha$ -Sasakian manifold  $M^n$  is said to be  $\varphi$ -projectively flat if

$$\varphi^2(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \quad (1.7)$$

for all vector fields  $X, Y, Z, W$  orthogonal to  $\xi$ , where  $P$  is the projective curvature tensor defined in (1.6).

In the present paper, we study Lorentzian  $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection. Motivated by the above studies in this paper we give the relation between the Levi-Civita connection and the quarter-symmetric metric connection on a Lorentzian  $\alpha$ -Sasakian manifold. We characterize quasi-projectively flat Lorentzian  $\alpha$ -Sasakian manifold with respect to the quarter-symmetric metric connection. Then we study  $\varphi$ -symmetric Lorentzian  $\alpha$ -Sasakian manifolds with respect to quarter-symmetric metric connection. We also study  $\varphi$ -projectively flat Lorentzian  $\alpha$ -Sasakian manifold with respect to the quarter-symmetric metric connection. Next we cultivate Lorentzian  $\alpha$ -Sasakian manifold with respect to the quarter-symmetric metric connection satisfying  $\tilde{P}.\tilde{S} = 0$ . Finally we give an example of 3-dimensional Lorentzian  $\alpha$ -Sasakian manifolds with respect to quarter-symmetric metric connection.

## 2 Preliminaries

A  $n(= 2m+1)$ -dimensional differentiable manifold  $M$  is said to be a Lorentzian  $\alpha$ -Sasakian manifold if it admits a  $(1, 1)$  tensor field  $\varphi$ , a contravariant vector field  $\xi$ , a covariant vector field  $\eta$  and Lorentzian metric  $g$  which satisfy the following conditions

$$\varphi^2 X = X + \eta(X)\xi, \quad (2.1)$$

$$\eta(\xi) = -1, \varphi\xi = 0, \eta(\varphi X) = 0, \quad (2.2)$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)$$

$$g(X, \xi) = \eta(X), \quad (2.4)$$

$$(\nabla_X \varphi)(Y) = \alpha\{g(X, Y)\xi + \eta(Y)X\} \quad (2.5)$$

$\forall X, Y \in \chi(M)$  and for smooth functions  $\alpha$  on  $M$ ,  $\nabla$  denotes the covariant differentiation with respect to Lorentzian metric  $g$  ([9], [17]).

For a Lorentzian  $\alpha$ -Sasakian manifold, it can be shown that ([9], [17]):

$$\nabla_X \xi = -\alpha\varphi X, \quad (2.6)$$

$$(\nabla_X \eta)(Y) = -\alpha g(\varphi X, Y) \quad (2.7)$$

for all  $X, Y \in \chi(M)$ . Further on a Lorentzian  $\alpha$ -Sasakian manifold, the following relations hold ([9])

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (2.8)$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \quad (2.9)$$

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \quad (2.10)$$

$$R(\xi, X)\xi = \alpha^2[X + \eta(X)\xi], \quad (2.11)$$

$$S(X, \xi) = S(\xi, X) = (n-1)\alpha^2\eta(X), \quad (2.12)$$

$$S(\xi, \xi) = -(n-1)\alpha^2, \quad (2.13)$$

$$Q\xi = (n-1)\alpha^2\xi, \quad (2.14)$$

where  $Q$  is the Ricci operator, i.e.

$$g(QX, Y) = S(X, Y), \quad (2.15)$$

$$S(\varphi X, \varphi Y) = S(X, Y) + (n-1)\alpha^2g(Y, Z). \quad (2.16)$$

If  $\nabla$  is the Levi-Civita connection manifold  $M$ , then quarter-symmetric metric connection  $\tilde{\nabla}$  in  $M$  is denoted by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi(X). \quad (2.17)$$

Now we will give the existence of the quarter-symmetric metric connection  $\tilde{\nabla}$  on a Lorentzian  $\alpha$ -Sasakian manifold  $M$ .

Let  $X, Y, Z$  be any vectors fields on a Lorentzian  $\alpha$ -Sasakian manifold  $M$  and let a connection  $\tilde{\nabla}$  is given by

$$\begin{aligned} 2g(\tilde{\nabla}_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) \\ &- g([Y, Z], X) + g([Z, X], Y) + g(\eta(Y)\varphi X \\ &- \eta(X)\varphi Y, Z) + g(\eta(X)\varphi Z - \eta(Z)\varphi X, Y) \\ &+ g(\eta(Y)\varphi Z - \eta(Z)\varphi Y, X). \end{aligned} \quad (2.18)$$

Then  $\tilde{\nabla}$  is a quarter-symmetric metric connection on  $M$ . The proof of this has been discussed on [8].

### 3 Curvature tensor and Ricci tensor of Lorentzian $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection

Let  $\tilde{R}(X, Y)Z$  and  $R(X, Y)Z$  be the curvature tensors with respect to the quarter-symmetric metric connection  $\tilde{\nabla}$  and with respect to the Riemannian connection  $\nabla$  respectively on a Lorentzian  $\alpha$ -Sasakian manifold  $M$ . A relation between the curvature tensors  $\tilde{R}(X, Y)Z$  and  $R(X, Y)Z$  on  $M$  is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + \alpha[g(\varphi X, Z)\varphi Y - g(\varphi Y, Z)\varphi X] \\ &+ \alpha\eta(Z)[\eta(Y)X - \eta(X)Y]. \end{aligned} \quad (3.1)$$

Also from (3.1), we obtain

$$\tilde{S}(X, Y) = S(X, Y) + \alpha[g(X, Y) + n\eta(X)\eta(Y)], \quad (3.2)$$

where  $\tilde{S}$  and  $S$  are the Ricci tensor with respect to  $\tilde{\nabla}$  and  $\nabla$  respectively. Contracting (3.2), we obtain,

$$\tilde{r} = r, \quad (3.3)$$

where  $\tilde{r}$  and  $r$  are the scalar curvature tensor with respect to  $\tilde{\nabla}$  and  $\nabla$  respectively. Also we have

$$\tilde{R}(\xi, X)Y = -\tilde{R}(X, \xi)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X] + \alpha\eta(Y)[X + \eta(X)\xi], \quad (3.4)$$

$$\eta(\tilde{R}(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (3.5)$$

$$\tilde{R}(X, Y)\xi = (\alpha^2 - \alpha)[\eta(Y)X - \eta(X)Y], \quad (3.6)$$

$$\tilde{S}(X, \xi) = \tilde{S}(\xi, X) = (n - 1)(\alpha^2 - \alpha)\eta(X), \quad (3.7)$$

$$\tilde{S}(\xi, \xi) = -(n - 1)(\alpha^2 - \alpha), \quad (3.8)$$

$$\tilde{Q}X = QX - \alpha(n - 1)X, \quad (3.9)$$

$$\tilde{Q}\xi = (n - 1)(\alpha^2 - \alpha)\xi, \quad (3.10)$$

$$\tilde{R}(\xi, X)\xi = (\alpha^2 - \alpha)[X + \eta(X)\xi], \quad (3.11)$$

#### 4 Quasi-projectively flat Lorentzian $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection

The projective curvature tensor  $\tilde{P}$  with respect to quarter-symmetric metric connection is defined by

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{n-1}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y] \quad (4.1)$$

for  $X, Y, Z \in \chi(M)$ , where  $\tilde{S}$  is the Ricci tensor of the manifold with respect to quarter-symmetric metric connection.

A Lorentzian  $\alpha$ -Sasakian manifold  $M^n$  is said to be quasi-projectively flat with respect to the quarter-symmetric metric connection if

$$g(\tilde{P}(\varphi X, Y)Z, \varphi W) = 0 \quad (4.2)$$

for all vector fields  $X, Y, Z, W$  orthogonal to  $\xi$ .

In view of the equation (4.1), we have

$$\begin{aligned} g(\tilde{P}(X, Y)Z, W) &= g(\tilde{R}(X, Y)Z, W) - \frac{1}{n-1}[\tilde{S}(Y, Z)g(X, W) \\ &\quad - \tilde{S}(X, Z)g(Y, W)]. \end{aligned} \quad (4.3)$$

Putting  $X = \varphi X$  and  $W = \varphi W$  in the above equation, we get

$$\begin{aligned} g(\tilde{P}(\varphi X, Y)Z, \varphi W) &= g(\tilde{R}(\varphi X, Y)Z, \varphi W) - \frac{1}{n-1}[\tilde{S}(Y, Z)g(\varphi X, \varphi W) \\ &\quad - \tilde{S}(\varphi X, Z)g(Y, \varphi W)]. \end{aligned} \quad (4.4)$$

Now assume that  $M^n$  is quasi-projectively flat with respect to quarter-symmetric metric connection. Then by virtue of equations (4.2) and (4.4), we have

$$g(\tilde{R}(\varphi X, Y)Z, \varphi W) = \frac{1}{n-1}[\tilde{S}(Y, Z)g(\varphi X, \varphi W) - \tilde{S}(\varphi X, Z)g(Y, \varphi W)], \quad (4.5)$$

Using equations (3.1) and (3.2) in above equation, we get

$$\begin{aligned} g(R(\varphi X, Y)Z, \varphi W) &= -\alpha[g(X, Z)g(Y, W) + g(X, Z)\eta(Y)\eta(W) \\ &\quad + g(Y, W)\eta(X)\eta(Z) + \eta(X)\eta(Y)\eta(Z)\eta(W) \\ &\quad - g(\varphi Y, Z)g(\varphi W, X)] - \alpha g(X, W)\eta(Y)\eta(Z) \\ &\quad - \alpha\eta(X)\eta(Y)\eta(Z)\eta(W) + \frac{1}{n-1}[S(Y, Z)g(\varphi X, \varphi W) \\ &\quad + \alpha\{g(Y, Z) + n\eta(Y)\eta(Z)\}g(\varphi X, \varphi W) \\ &\quad - S(\varphi X, Z)g(\varphi W, Y) - \alpha g(\varphi X, Z)g(\varphi W, Y)]. \end{aligned} \quad (4.6)$$

Let  $\{e_1, e_2, \dots, e_{n-1}, \xi\}$  be a local orthonormal basis of vector fields in  $M^n$ . Then  $\{\varphi e_1, \varphi e_2, \dots, \varphi e_{n-1}, \xi\}$  is also a local orthonormal basis of  $M^n$ . Putting  $X = W = e_i$  in (4.6) and taking summation over

i,  $1 \leq i \leq n-1$ , we get

$$\begin{aligned}
\sum_{i=1}^{n-1} g(R(\varphi e_i, Y)Z, \varphi e_i) &= -\alpha \sum_{i=1}^{n-1} [g(e_i, Z)g(Y, e_i) + g(e_i, Z)\eta(Y)\eta(e_i) \\
&\quad + g(Y, e_i)\eta(e_i)\eta(Z) + \eta(e_i)\eta(Y)\eta(Z)\eta(e_i) \\
&\quad - g(\varphi Y, Z)g(\varphi e_i, e_i)] - \sum_{i=1}^{n-1} [\alpha g(e_i, e_i)\eta(Y)\eta(Z) \\
&\quad - \alpha \eta(e_i)\eta(Y)\eta(Z)\eta(e_i)] + \frac{1}{n-1} \sum_{i=1}^{n-1} [S(Y, Z) \\
&\quad g(\varphi e_i, \varphi e_i) + \alpha \{g(Y, Z) + n\eta(Y)\eta(Z)\}g(\varphi e_i, \varphi e_i) \\
&\quad - S(\varphi e_i, Z)g(\varphi e_i, Y) - \alpha g(\varphi e_i, Z)g(\varphi e_i, Y)]. \tag{4.7}
\end{aligned}$$

Also,

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, Y)Z, \varphi e_i) = S(Y, Z) + g(Y, Z), \tag{4.8}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, Z)g(\varphi e_i, Y) = S(Y, Z), \tag{4.9}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n-1 \tag{4.10}$$

and

$$\sum_{i=1}^{n-1} g(\varphi e_i, Y)g(\varphi e_i, Z) = g(Y, Z). \tag{4.11}$$

Hence, by virtue of equations (4.8), (4.9), (4.10) and (4.11), the equation (4.7) takes the form

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z) + c\Phi(Y, Z), \tag{4.12}$$

where  $a = (1-n)(\alpha^2 + 1)$ ,  $b = (n-1)\alpha$ ,  $c = \alpha(n-1)\beta$  and

$$\beta = \text{trace}\Phi, \Phi(Y, Z) = g(\varphi Y, Z).$$

Hence we can state the following

**Theorem 4.1.** *A quasi-projectively flat Lorentzian  $\alpha$ -Sasakian manifold with respect to a quarter-symmetric metric connection is an generalized  $\eta$ -Einstein manifold.*

## 5 $\varphi$ -symmetric Lorentzian $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection

A Lorentzian  $\alpha$ -Sasakian manifold  $M^n$  is said to be  $\varphi$ -symmetric with respect to the quarter-symmetric metric connection if

$$\varphi^2((\tilde{\nabla}_W \tilde{R})(X, Y)Z) = 0 \quad (5.1)$$

for arbitrary vector fields  $X, Y, Z, W$ .

From the equation (2.17) and (3.1), we have

$$(\tilde{\nabla}_W \tilde{R})(X, Y)Z = (\nabla_W \tilde{R})(X, Y)Z + \eta(\tilde{R}(X, Y)Z)\varphi W. \quad (5.2)$$

Let us consider a  $\varphi$ -symmetric Lorentzian  $\alpha$ -Sasakian manifolds with respect to quarter-symmetric metric connection. Then by virtue of (2.1) and (5.1) we have

$$((\tilde{\nabla}_W \tilde{R})(X, Y)Z) + \eta((\tilde{\nabla}_W \tilde{R})(X, Y)Z)\xi = 0. \quad (5.3)$$

From which it follows that

$$g((\tilde{\nabla}_W \tilde{R})(X, Y)Z, U) + \eta((\tilde{\nabla}_W \tilde{R})(X, Y)Z)g(\xi, U) = 0. \quad (5.4)$$

Let  $\{e_i\}$ ,  $i = 1, 2, \dots, n$  be an orthonormal basis of the tangent space at any point of the manifold. Then putting  $X = U = e_i$  in (5.4) and taking summation over  $i$ ,  $1 \leq i \leq n$ , we have

$$(\tilde{\nabla}_W \tilde{S})(Y, Z) + \sum_{i=1}^n \eta((\tilde{\nabla}_W \tilde{R})(e_i, Y)Z)\eta(e_i) = 0. \quad (5.5)$$

The second term of (5.5) by putting  $Z = \xi$  takes the form

$$\eta((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi)\eta(e_i) = g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi)g(e_i, \xi). \quad (5.6)$$

By using (2.16) and (5.2), we can write

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = g((\nabla_W \tilde{R})(e_i, Y)\xi, \xi) + \eta(\tilde{R}(e_i, Y)\xi)\varphi W. \quad (5.7)$$

After some calculations, from (5.7) we have

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = g((\nabla_W \tilde{R})(e_i, Y)\xi, \xi). \quad (5.8)$$

In Lorentzian  $\alpha$ -Sasakian manifold, we have

$$g((\nabla_W \tilde{R})(e_i, Y)\xi, \xi) = 0.$$

So from (5.8) we get

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = 0. \quad (5.9)$$



By replacing  $Z = \xi$  in (5.5) and using (5.9), we get

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = 0. \quad (5.10)$$

We know that

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi). \quad (5.11)$$

Now using (2.6), (2.12), (2.17) and (3.7), we obtain

$$\begin{aligned} (\tilde{\nabla}_W \tilde{S})(Y, \xi) &= -(n-1)(\alpha^2 - \alpha)\alpha g(Y, \varphi W) + (\alpha+1)[S(Y, \varphi W) \\ &\quad + \alpha g(Y, \varphi W)]. \end{aligned} \quad (5.12)$$

Applying (5.11) in (5.10), we obtain

$$S(Y, \varphi W) = g(Y, \varphi W)[(n-1)\alpha^2 \frac{\alpha-1}{\alpha+1} - \alpha]. \quad (5.13)$$

Replacing  $W$  by  $\varphi W$  we get

$$\begin{aligned} S(Y, W) &= g(Y, W)[(n-1)\alpha^2 \frac{\alpha-1}{\alpha+1} - \alpha] \\ &\quad - \frac{\alpha^3(n+1) + 3n\alpha^2 + \alpha}{\alpha+1} \eta(Y)\eta(W). \end{aligned} \quad (5.14)$$

Contracting (5.14), we get

$$r = n[(n-1)\alpha^2 \frac{\alpha-1}{\alpha+1} - \alpha] + \frac{\alpha^3(n+1) + 3n\alpha^2 + \alpha}{\alpha+1}. \quad (5.15)$$

This leads to the following theorem:

**Theorem 5.1.** *Let  $M$  be a  $\varphi$ -symmetric Lorentzian  $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection  $\tilde{\nabla}$ . Then the manifold has a scalar curvature  $r = n[(n-1)\alpha^2 \frac{\alpha-1}{\alpha+1} - \alpha] + \frac{\alpha^3(n+1) + 3n\alpha^2 + \alpha}{\alpha+1}$  with respect to Levi-Civita connection  $\nabla$  of  $M$  provided  $\alpha \neq -1$ .*

## 6 $\varphi$ -projectively flat Lorentzian $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection

A Lorentzian  $\alpha$ -Sasakian manifold  $M^n$  is said to  $\varphi$ -projectively flat with respect to the quarter-symmetric metric connection if

$$\varphi^2(\tilde{P}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \quad (6.1)$$

for all vector fields  $X, Y, Z, W$  orthogonal to  $\xi$ , where  $\tilde{P}$  is the projective curvature tensor defined in (4.1).

Let  $M^n$  be a  $\varphi$ -projectively flat Lorentzian  $\alpha$ -Sasakian manifold with respect to the quarter-symmetric metric connection. It is easy to see that  $\varphi^2(\tilde{P}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0$  holds if and only if

$$g(\tilde{P}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \quad (6.2)$$

for any  $X, Y, Z, W \in \chi(M)$ .

Putting  $Y = \varphi Y$  and  $Z = \varphi Z$  in (4.4), we get

$$\begin{aligned} g(\tilde{P}(\varphi X, \varphi Y)\varphi Z, \varphi W) &= g(\tilde{R}(\varphi X, \varphi Y)\varphi Z, \varphi W) - \frac{1}{n-1}[\tilde{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- \tilde{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \quad (6.3)$$

Using the equation (6.3) in the equation (6.2), we obtain

$$\begin{aligned} g(\tilde{R}(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{n-1}[\tilde{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- \tilde{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \quad (6.4)$$

Now using the equation (3.1) and (3.2), we obtain

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= -\alpha[g(X, \varphi Z)g(Y, \varphi W) - g(Y, \varphi Z)g(X, \varphi W)] \\ &+ \frac{1}{n-1}[S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) + \alpha g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - \alpha g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \quad (6.5)$$

Let  $\{e_1, e_2, \dots, e_{n-1}, \xi\}$  be a local orthonormal basis of vector fields in  $M^n$ . Then  $\{\varphi e_1, \varphi e_2, \dots, \varphi e_{n-1}, \xi\}$  is also a local orthonormal basis of  $M_n$ . Putting  $X = W = e_i$  in (6.5) and taking summation over  $i$ ,  $1 \leq i \leq n-1$ , we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= -\alpha \sum_{i=1}^{n-1} [g(X, \varphi Z)g(Y, \varphi W) - g(Y, \varphi Z)g(X, \varphi W)] \\ &+ \frac{1}{n-1} \sum_{i=1}^{n-1} [S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &+ \alpha g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) \\ &- \alpha g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \quad (6.6)$$

Hence, by virtue of equations (4.8), (4.9), (4.10) and (4.11), the equation (6.6) takes the form

$$S(\varphi Y, \varphi Z) = g(\varphi Y, \varphi Z)\left[\alpha - 1 - \frac{\alpha}{n-1}\right] + \alpha(\beta - 1)\Phi(Y, Z), \quad (6.7)$$

where  $\beta = \text{trace}\Phi$ ,  $\Phi(Y, Z) = g(\varphi Y, Z)$ . Using the equation (2.3) and (2.16), we obtain

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z) + c\Phi(Y, Z), \quad (6.8)$$

where  $a = [\alpha - 1 - \frac{\alpha}{n-1} - (n-1)\alpha^2]$ ,  $b = [\alpha - 1 - \frac{\alpha}{n-1}]$  and  $c = \alpha(\beta - 1)$ . Hence we can state the following

**Theorem 6.1.** *A  $\varphi$ -projectively flat Lorentzian  $\alpha$ -Sasakian manifold with respect to a quarter-symmetric metric connection is an generalized  $\eta$ -Einstein manifold.*

## 7 Lorentzian $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection satisfying $\tilde{P}.\tilde{S} = 0$

A Lorentzian  $\alpha$ -Sasakian manifold with respect to the quarter-symmetric metric connection satisfying

$$(\tilde{P}(X, Y).\tilde{S})(Z, U) = 0, \quad (7.1)$$

where  $\tilde{S}$  is the Ricci tensor with respect to a quarter-symmetric metric connection. Then, we have

$$\tilde{S}(\tilde{P}(X, Y)Z, U) + \tilde{S}(Z, \tilde{P}(X, Y)U) = 0. \quad (7.2)$$

Putting  $X = \xi$  in the equation (7.2), we have

$$\tilde{S}(\tilde{P}(\xi, Y)Z, U) + \tilde{S}(Z, \tilde{P}(\xi, Y)U) = 0. \quad (7.3)$$

In view of the equation (4.1), we have

$$\tilde{P}(\xi, Y)Z = \tilde{R}(\xi, Y)Z - \frac{1}{n-1}[\tilde{S}(Y, Z)\xi - \tilde{S}(\xi, Z)Y] \quad (7.4)$$

for  $X, Y, Z \in \chi(M)$ .

Using equations (3.4) and (3.7) in the equation (7.4), we get

$$\begin{aligned} \tilde{P}(\xi, Y)Z &= \alpha^2[g(Y, Z)\xi - \eta(Z)Y] + \alpha\eta(Z)[Y + \eta(Y)\xi] \\ &- \frac{1}{n-1}[S(Y, Z)\xi + \alpha\{g(Y, Z) + n\eta(Y)\eta(Z)\}\xi] \\ &- (\alpha^2 - \alpha)(n-1)\eta(Z)Y. \end{aligned} \quad (7.5)$$

Now using the equation (7.5) and putting  $U = \xi$  in the equation (7.3) and using the equations (2.2), (2.12), we get

$$\begin{aligned} S(Y, Z)[\alpha^2 &+ \frac{\alpha}{n-1} - \frac{\alpha n}{n-1}] + g(Y, Z)[- \alpha^4(n-1) + \frac{\alpha^2}{n-1} + n\alpha^3 - \frac{\alpha^2 n}{n-1}] \\ &+ \eta(Y)\eta(Z)[(\alpha^2 - \alpha)(n-1)\alpha^2 + \alpha(\alpha^2 - \alpha^2)] \\ &+ 2\alpha^3 n - n\alpha^4] = 0. \end{aligned} \quad (7.6)$$

This gives

$$S(Y, Z) = a g(Y, Z) + b \eta(Y)\eta(Z), \quad (7.7)$$

where  $a = \alpha[n\alpha - (\alpha + 1)]$  and  $b = 2\alpha^2 + \frac{n\alpha^2}{n-1}$ . In view of above discussions we can state the following theorem:

**Theorem 7.1.** *A  $n$ -dimensional Lorentzian  $\alpha$ -Sasakian manifold with a quarter-symmetric metric connection satisfying  $\tilde{P}.\tilde{S} = 0$ . is an  $\eta$ -Einstein manifold.*

## 8 Example of 3-dimensional Lorentzian $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection

We consider a 3-dimensional manifold  $M = \{(x, y, u) \in R^3\}$ , where  $(x, y, u)$  are the standard coordinates of  $R^3$ . Let  $e_1, e_2, e_3$  be the vector fields on  $M^3$  given by

$$e_1 = e^u \frac{\partial}{\partial x}, \quad e_2 = e^u \frac{\partial}{\partial y}, \quad e_3 = e^u \frac{\partial}{\partial u}.$$

Clearly,  $\{e_1, e_2, e_3\}$  is a set of linearly independent vectors for each point of  $M$  and hence a basis of  $\chi(M)$ . The Lorentzian metric  $g$  is defined by

$$\begin{aligned} g(e_1, e_2) &= g(e_2, e_3) = g(e_1, e_3) = 0, \\ g(e_1, e_1) &= 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1. \end{aligned}$$

Let  $\eta$  be the 1-form defined by  $\eta(Z) = g(Z, e_3)$  for any  $Z \in \chi(M)$  and the  $(1, 1)$  tensor field  $\varphi$  is defined by

$$\varphi e_1 = -e_1, \quad \varphi e_2 = -e_2, \quad \varphi e_3 = 0.$$

From the linearity of  $\varphi$  and  $g$ , we have

$$\begin{aligned} \eta(e_3) &= -1, \\ \varphi^2 X &= X + \eta(X)e_3 \end{aligned}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any  $X \in \chi(M)$ . Then for  $e_3 = \xi$ , the structure  $(\varphi, \xi, \eta, g)$  defines a Lorentzian paracontact structure on  $M$ .

Let  $\nabla$  be the Levi-Civita connection with respect to the Lorentzian metric  $g$ . Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 e^{-u}, \quad [e_2, e_3] = e_2 e^{-u}.$$

Koszul's formula is defined by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$

$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Then from above formula we can calculate the followings:

$$\begin{aligned}\nabla_{e_1} e_1 &= -e_3 e^u, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1 e^u, \\ \nabla_{e_2} e_1 &= 0, \quad \nabla_{e_2} e_2 = -e_3 e^u, \quad \nabla_{e_2} e_3 = -e_2 e^u, \\ \nabla_{e_3} e_1 &= 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.\end{aligned}$$

From the above calculations, we see that the manifold under consideration satisfies  $\eta(\xi) = -1$  and  $\nabla_X \xi = -\alpha \varphi X$  for  $\alpha = -e^u$ .

Hence the structure  $(\varphi, \xi, \eta, g)$  is a Lorentzian  $\alpha$ -Sasakian structure.

Using (2.17), we find  $\tilde{\nabla}$ , the quarter-symmetric metric connection on  $M$  following:

$$\begin{aligned}\tilde{\nabla}_{e_1} e_1 &= -e_3 e^u, \quad \tilde{\nabla}_{e_1} e_2 = 0, \quad \tilde{\nabla}_{e_1} e_3 = e_1(1 - e^u), \\ \tilde{\nabla}_{e_2} e_1 &= 0, \quad \tilde{\nabla}_{e_2} e_2 = -e_3 e^u, \quad \tilde{\nabla}_{e_2} e_3 = e_2(1 - e^u), \\ \tilde{\nabla}_{e_3} e_1 &= 0, \quad \tilde{\nabla}_{e_3} e_2 = 0, \quad \tilde{\nabla}_{e_3} e_3 = 0.\end{aligned}$$

Using (1.2), the torsion tensor  $T$ , with respect to quarter-symmetric metric connection  $\tilde{\nabla}$  as follows:

$$\begin{aligned}\tilde{T}(e_i, e_i) &= 0, \quad \forall i = 1, 2, 3, \\ \tilde{T}(e_1, e_2) &= 0, \quad \tilde{T}(e_1, e_3) = e_1, \quad \tilde{T}(e_2, e_3) = e_2.\end{aligned}$$

Also,

$$(\tilde{\nabla}_{e_1} g)(e_2, e_3) = 0, \quad (\tilde{\nabla}_{e_2} g)(e_3, e_1) = 0, \quad (\tilde{\nabla}_{e_3} g)(e_1, e_2) = 0.$$

Thus  $M$  is Lorentzian  $\alpha$ -Sasakian manifold with quarter-symmetric metric connection  $\tilde{\nabla}$ .

By using the above results, we can easily obtain the components of the curvature tensor as follows:

$$\begin{aligned}R(e_1, e_3)e_3 &= -e_1 \alpha^2, \quad R(e_2, e_1)e_1 = e_2 \alpha^2, \quad R(e_2, e_3)e_3 = -e_2 \alpha^2, \\ R(e_3, e_1)e_1 &= e_3 \alpha^2, \quad R(e_3, e_2)e_2 = e_3 \alpha^2, \quad R(e_1, e_2)e_3 = 0, \\ R(e_2, e_3)e_2 &= -e_3 \alpha^2, \quad R(e_1, e_2)e_2 = e_1 \alpha^2\end{aligned}$$

and

$$\begin{aligned}\tilde{R}(e_1, e_3)e_3 &= e_1(\alpha - \alpha^2), \quad \tilde{R}(e_2, e_1)e_1 = e_2(\alpha^2 - \alpha), \\ \tilde{R}(e_2, e_3)e_3 &= e_2(\alpha - \alpha^2), \quad \tilde{R}(e_3, e_1)e_1 = e_3 \alpha^2, \\ \tilde{R}(e_3, e_2)e_2 &= e_3 \alpha^2, \quad \tilde{R}(e_1, e_2)e_3 = 0, \quad \tilde{R}(e_2, e_3)e_2 = -e_3 \alpha^2,\end{aligned}$$

$$\tilde{R}(e_1, e_2)e_2 = e_1(\alpha^2 - \alpha).$$

Using the expressions of the curvature tensors, we find the values of the Ricci tensors as follows:

$$S(e_1, e_1) = 0, \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = -2\alpha^2,$$

$$S(e_1, e_2) = 0, \quad S(e_2, e_3) = 0, \quad S(e_1, e_3) = 0$$

and

$$\tilde{S}(e_1, e_1) = \alpha, \quad \tilde{S}(e_2, e_2) = \alpha, \quad \tilde{S}(e_3, e_3) = 2(\alpha - \alpha^2),$$

$$\tilde{S}(e_1, e_2) = 0, \quad \tilde{S}(e_2, e_3) = 0, \quad \tilde{S}(e_1, e_3) = 0.$$

Let  $\Phi(X, Y)$  be a  $(0, 2)$  tensor. Then we have,

$$\Phi(X, Y) = g(X, \varphi Y).$$

Then

$$\Phi(e_1, e_1) = g(e_1, \varphi e_1) = g(e_1, -e_1) = -1,$$

$$\Phi(e_2, e_2) = g(e_2, \varphi e_2) = g(e_2, -e_2) = -1,$$

and

$$\Phi(e_3, e_3) = g(e_3, \varphi e_3) = g(e_1, 0) = 0.$$

If we take the scalars  $a_1, b_1$  as follows:

$$a_1 = \alpha \text{ and } b_1 = 3\alpha - 2\alpha^2,$$

then we have

$$\tilde{S}(e_1, e_1) = a_1 g(e_1, e_1) + b_1 \eta(e_1) \eta(e_1),$$

$$\tilde{S}(e_2, e_2) = a_1 g(e_2, e_2) + b_1 \eta(e_2) \eta(e_2),$$

$$\tilde{S}(e_3, e_3) = a_1 g(e_3, e_3) + b_1 \eta(e_3) \eta(e_3).$$

Thus the manifold under consideration is a  $\eta$ -Einstein Lorentzian  $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection. Again, if we take another scalars

$$a_2 = \alpha + 1, \quad b_2 = 3\alpha - 2\alpha^2 + 1, \quad f = 1,$$

then we have

$$\tilde{S}(e_1, e_1) = a_2 g(e_1, e_1) + b_2 \eta(e_1) \eta(e_1) + f \Phi(e_1, e_1),$$

$$\begin{aligned}\tilde{S}(e_2, e_2) &= a_2g(e_2, e_2) + b_2\eta(e_2)\eta(e_2) + f\Phi(e_2, e_2), \\ \tilde{S}(e_3, e_3) &= a_2g(e_3, e_3) + b_2\eta(e_3)\eta(e_3) + f\Phi(e_3, e_3).\end{aligned}$$

So, the manifold becomes generalized  $\eta$ -Einstein Lorentzian  $\alpha$ -Sasakian manifold with respect to quarter-symmetric metric connection.

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